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Convergence Rates of α-Stable Difference Methods

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1. INTRODUCTION

The purpose of this note is to prove a quantitative Lax type theorem expressed in terms of α-stability and α-well-posedness. Here the term "quantitative" means that in the consistency hypothesis upon the difference method a rate of convergence is prescribed and that in the convergence result a rate is obtained corresponding to the degree of smoothness of the initial value. Theorems of this sort have been established in a general Banach space setting by Butzer, Dickmeis, Nessel, and others [2, 4, 5, 7]. For specific L_p spaces such theorems were proven by Peetre and Thomée [10].

Our objective here is to work with α-well-posedness for $0 \leqslant \alpha < 1$, instead of the usual (strong) well-posedness, that is, to admit initial value problems

$$\frac{d}{dt} u = Au \quad (t > 0), \qquad u(0) = f \in X, \tag{1.1}$$

on a Banach space X, with the closed linear operator A forming the infinitesimal generator of a semigroup $\{E(t); t > 0\}$ *of growth order* α in the sense of Da Prato [6]. Essentially, the latter property requires that the operator norm of $E(t)$ satisfies

$$\|E(t)\| \leqslant M t^{-\alpha} e^{\omega t} \qquad (t > 0). \tag{1.2}$$

For $\alpha > 0$ this is a weaker property than strong well-posedness, to which it reduces for $\alpha = 0$. Examples of initial value problems which are α-well-posed for some $\alpha > 0$ but not 0-well-posed are frequently met among systems of differential equations of the form (1.1) when the symbol of A is not a normal matrix (cf. [8]; there also is given a characterization of α – well-posedness analogous to the Kreiss theorem).

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To α-well-posedness there corresponds the property of *α-stability* for a difference scheme $\{E_k; 0 < k \leqslant k_0\}$ of bounded linear operators on X, depending continuously on k. This requires the nth iterate of E_k to satisfy

$$\|E_k^n\| \leqslant C(nk)^{-\alpha} e^{Knk} \qquad (0 < k \leqslant k_0, n \in \mathbb{N}), \tag{1.3}$$

for certain constants C, K. Again see [8] for a characterization.

In Section 2 a quantitative Lax-type theorem on a general Banach space X will be given, and in Section 3 this will be made more precise by specializing X to the Lebesgue space L_p^N.

2. General Theorem

A difference scheme is said to be consistent with (1.1) of order $\varphi(k)$ on some linear subspace D of X if, for each $T > 0$, there is a constant C_0 such that, for each $g \in D$,

$$\| [E_k - E(k)] E(t) g\| \leqslant C_0 k\varphi(k) \| g\|_D \quad (0 < k \leqslant k_0, 0 \leqslant t \leqslant T). \tag{2.1}$$

Here $E(0)$ denotes the identity operator, $\|\cdot\|_D$ is some norm on D, and φ is an increasing function with $\varphi(x) \to 0$ as $x \to 0+$.

We further use the (Peetre) K-functional, which is defined by

$$K(t, f; X, D) = \inf_{g \in D} \{\|f - g\| + t \| g\|_D\}, \tag{2.2}$$

for $f \in X$, $t > 0$.

Our first result is

Theorem 1. *Let the initial value problem* (1.1) *be α-well-posed for some $\alpha \in [0, 1)$, and let $\{E_k; 0 < k \leqslant k_0\}$ be a difference scheme which is consistent with* (1.1) *of order $\varphi(k)$ on a subspace D of X. The following assertions are equivalent*:

(a) *the difference scheme is α-stable*;

(b) *for each $T > 0$ there is a constant C_1 such that*

$$\|E_k^n f - E(nk) f\| \leqslant C_1 (nk)^{-\alpha} K(nk\varphi(k), f; X, D)$$

for $0 < k \leqslant k_0$, $nk \leqslant T$, and $f \in X$;

(c) *for arbitrary $T > 0$ there is a constant C_2 such that*

$$\begin{aligned}\|E_k^n f - E(nk) f\| &\leqslant C_2 (nk)^{-\alpha} \|f\|, & f \in X,\\ &\leqslant C_2 (nk)^{1-\alpha} \varphi(k) \|f\|_D, & f \in D,\end{aligned}$$

for $0 < k \leqslant k_0$, $nk \leqslant T$.

Proof. Assuming (a) to hold, let $g \in D$. Using (1.3), (2.1), it follows for $0 < k \leqslant k_0$, $n \geqslant 2$, and $nk \leqslant T$ that

$$\begin{aligned}
\|E_k^n g - E(nk)\, g\| &\leqslant \sum_{j=0}^{n-1} \|E_k^{n-j-1}\| \, \|[E_k - E(k)]\, E(jk)\, g\| \\
&\leqslant C_0 k\varphi(k) \|g\|_D \left\{ 1 + Ck^{-\alpha} \sum_{j=0}^{n-2} (n-j-1)^{-\alpha} \right\} \\
&\leqslant C_0 k\varphi(k) \|g\|_D \left\{ 1 + Ck^{-\alpha} \frac{n^{1-\alpha}}{1-\alpha} \right\} \\
&\leqslant C_0'(nk)^{1-\alpha}\, \varphi(k) \|g\|_D,
\end{aligned}$$

for some constant C_0' which is independent of n, k, and g. The case $n = 1$ being trivial, property (6) follows by observing that

$$\begin{aligned}
\|E_k^n f - E(nk) f\| &\leqslant \inf_{g \in D} \{\|(E_k^n - E(nk))(f-g)\| + \|E_k^n g - E(nk)\, g\|\} \\
&\leqslant \inf_{g \in D} \{(Ce^{KT} + Me^{\omega T})(nk)^{-\alpha} \|f-g\| \\
&\qquad + C_0'(nk)^{1-\alpha}\, \varphi(k) \|g\|_D\} \\
&\leqslant C_1(nk)^{-\alpha} K(nk\varphi(k), f; X, D).
\end{aligned}$$

The implication (b) $\Rightarrow$ (c) follows from the very definition (2.2). Let (c) be satisfied. Inserting (1.2) into $\|E_k^n - E(nk)\| \leqslant C_2(nk)^{-\alpha}$ we have

$$\|E_k^n\| \leqslant (C_2 + Me^{\omega T})(nk)^{-\alpha} \qquad (0 < k \leqslant k_0,\ nk \leqslant T),$$

which implies (1.3), that is, (a), and the proof is complete. ∎

It may be noted that in (b) and (c) the estimate on D can be replaced by

$$\|E_k^n f - E(nk) f\| \leqslant C_1 k\varphi(k)\{1 + (nk)^{-\alpha}(n-1)\} \|f\|_D,$$

which is more precise in case $n = 1$ (cf. (2.1)).

3. Lebesgue Space Case

Let L_p^N denote the N fold Cartesian product of $L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, with norm

$$\|f\|_p = \sum_{j=1}^{N} \|f_j\|_p \qquad (f = (f_1, ..., f_N) \in L_p^N), \tag{3.1}$$

where, for an $f_j \in L_p(\mathbb{R}^d)$,

$$\|f_j\|_p = \left(\int_{\mathbb{R}^d} |f_j(x)|^p \, dx\right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$= \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f_j(x)|, \qquad p = \infty.$$

By $W^N_{p,m}$ we denote the N fold product of Sobolev spaces

$$W^N_{p,m} = \sum_{j=1}^{N} W_{p,m_j}(\mathbb{R}^d),$$

where $m = (m_1, ..., m_N)$ and $m_j \in \mathbb{P} = \{0, 1, 2, ...\}$ for each j, the norms being defined by

$$\|f\|_{p,m} = \sum_{j=1}^{N} \|f_j\|_{p,m_j} \qquad (f = (f_1, ..., f_N) \in W^N_{p,m}), \tag{3.2}$$

with

$$\|f_j\|_{p,m_j} = \sum_{|r| \leqslant m_j} \|D^r f\|_p \qquad (f_j \in W_{p,m_j}(\mathbb{R}^d)).$$

Here D^r is the differential operator $\partial^{|r|}/\partial x_1^{r_1} \cdots \partial x_d^{r_d}$ and $|r| = r_1 + \cdots + r_d$.

We further need the Besov spaces

$$B_p^{s,\infty,N} = \prod_{j=1}^{N} B_p^{s_j,\infty}(\mathbb{R}^d) \qquad (s = (s_1, ..., s_N), s_j > 0),$$

with norms

$$\|f\|_{B_p^{s,\infty,N}} = \|f\|_p + \sum_{j=1}^{N} \sup_{t>0} t^{-s_j} \omega_{m_j}(t, f_j)_p \qquad (0 < s_j < m_j),$$
$$\omega_{m_j}(r, f_j)_p = \sup_{0 < |h| \leqslant t} \left\| \sum_{l=0}^{m_j} \binom{m_j}{l} (-1)^{m_j - l} f_j(\cdot + lh) \right\|_p \qquad (h \in \mathbb{R}^d). \tag{3.3}$$

These are intermediate between L_p^N and $W^N_{p,m}$, that is,

$$B_p^{s,\infty,N} = \prod_{j=1}^{N} (L_p(\mathbb{R}^d), W_{p,m_j}(\mathbb{R}^d))_{s_j/m_j,\infty} \qquad (0 < s_j < m_j),$$

cf. [3, Sect. 4.3.1] for details.

The following special case of Theorem 1 can be considered as an extension to $\alpha > 0$ of a result of Peetre–Thomée [10], cf. also Brenner–Thomée–Wahlbin [1; Sect. 3.3].

THEOREM 2. *Let the initial value problem* (1.1) *be α-well-posed on L_p^N for some $\alpha \in [0, 1)$, and let $\{E_k; 0 < k \leqslant k_0\}$ be a difference scheme which is α-stable on L_p^N and consistent with* (1.1) *of order $\varphi(k)$ on $W_{p,m}^N$, for some multi-index m.*

Given $s = (s_1, ..., s_N)$ with $0 < s_j < m_j$ and $T > 0$, there is a constant C_3 such that, for each $f \in B_p^{s,\infty,N}$, $0 < k \leqslant k_0$, $nk \leqslant T$, one has

$$\|E_k^n f - E(nk) f\|_p \leqslant C_3 \max_{1 \leqslant j \leqslant N} \{(nk)^{(s_j/m_j) - \alpha}\, \varphi(k)^{s_j/m_j}\}\, \|f\|_{B_p^{s,\infty,N}}.$$

Proof. By Theorem 1 we have for each $f \in L_p^N$,

$$\|E_k^n f - E(nk) f\|_p \leqslant C_1 (nk)^{-\alpha} K(nk\varphi(k), f; L_p^N, W_{p,m}^N).$$

In view of [3; (4.3.4)] (for $p = \infty$, see also [9]) and (3.1)–(3.3), there is a constant C_m which does not depend on t, such that

$$\begin{aligned} K(t, f; L_p^N, W_{p,m}^N) &= \inf_{g_j \in W_{p,m_j}(\mathbb{R}^d)} \left\{ \sum_{j=1}^N \|f_j - g_j\|_p + t \sum_{j=1}^N \|g_j\|_{p,m_j} \right\} \\ &= \sum_{j=1}^N K(t, f_j; L_p(\mathbb{R}^d), W_{p,m_j}(\mathbb{R}^d)) \\ &\leqslant C_m \sum_{j=1}^N [\min(1, t) \|f_j\|_p + \omega_{m_j}(t^{1/m_j}, f_j)_p] \\ &\leqslant C_m \sum_{j=1}^N [\min(1, t) + t^{s_j/m_j}] \|f\|_{B_p^{s,\infty,N}} \\ &\leqslant N C_m \max_{1 \leqslant j \leqslant N} [\min(1, t) + t^{s_j/m_j}] \|f\|_{B_p^{s,\infty,N}} \\ &\leqslant 2 N C_m \max_{1 \leqslant j \leqslant N} t^{s_j/m_j} \|f\|_{B_p^{s,\infty,N}} \qquad (t > 0). \end{aligned}$$

Setting $t = nk\varphi(k)$ the assertion follows. ∎

We finally note that Theorem 2 might also be written as an equivalence theorem if one adds the case $s_j = 0$, where $\|f\|_{B_p^{s,\infty,N}}$ has to be replaced by $\|f\|_p$.

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